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On the shape of spectra for non-self-adjoint periodic Schrödinger operators

Kwang C Shin

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

E-mail: kcshin@math.missouri.edu

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Abstract

The spectra of the Schrödinger operators with periodic potentials are studied. When the potential is real and periodic, the spectrum consists of, at most, countably many line segments (energy bands) on the real line, while when the potential is complex and periodic, the spectrum consists of, at most, countably many analytic arcs in the complex plane. In some recent papers, such operators with complex $\mathcal{PT}$-symmetric periodic potentials have been studied. In particular, the authors argued that some energy bands would appear and disappear under perturbations. Here, we show that the appearance and disappearance of such energy bands imply existence of nonreal spectra. This is a consequence of a more general result, describing the local shape of the spectrum.

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In recent papers [1–5], the appearance and disappearance of real energy bands for some complex $\mathcal{PT}$-symmetric periodic potentials under perturbations have been reported. In this paper, we show that the appearance and disappearance of such real energy bands imply the existence of nonreal band spectra.

We begin by introducing some facts on Floquet theory and the associated Hill operators H. Consider the Schrödinger equation

$$-\psi_{xx}(E,x) + V(x)\psi(E,x) = E\psi(E,x), \qquad x \in \mathbb{R}, \tag{1}$$

where $E \in \mathbb{C}$ and $V \in L^1_{loc}(\mathbb{R})$ is a complex-valued periodic function of period $\omega > 0$. The *Hill operator* H in $L^2(\mathbb{R})$ associated with (1) is defined by

$$(Hf)(x) = -f_{xx}(x) + V(x)f(x),$$
$$f \in \mathrm{dom}(H) = \{g \in L^2(\mathbb{R})\colon g, g_x \in AC_{loc};\ (-g_{xx} + Vg) \in L^2(\mathbb{R})\}.$$

Then H is a densely defined closed operator in $L^2(\mathbb{R})$ (see e.g. chapter 5 in [6]).

For each $E \in \mathbb{C}$, there exists a fundamental system of solutions $\phi_1(E, x)$, $\phi_2(E, x)$ of (1) such that

$$\phi_1(E,0) = 1, \qquad \phi_{1x}(E,0) = 0,$$

$$\phi_2(E,0) = 0, \qquad \phi_{2x}(E,0) = 1.$$

The *monodromy matrix* $M(E)$ of (1) is defined by

$$M(E) = \begin{pmatrix} \phi_1(E,\omega) & \phi_2(E,\omega) \\ \phi_{1x}(E,\omega) & \phi_{2x}(E,\omega) \end{pmatrix}.$$

Then $\rho_1(E)\rho_2(E) = 1$, where $\rho_1(E)$ and $\rho_2(E)$ are the eigenvalues (the *Floquet multipliers*) of $M(E)$ (see e.g. equation (1.2.4) in [6]). The *Floquet discriminant* $\Delta(E)$ is defined by half of the trace of $M(E)$, i.e. by

$$\Delta(E) = \frac{1}{2}\left(\phi_1(E,\omega) + \phi_{2x}(E,\omega)\right) = \frac{1}{2}\left(\rho(E) + \frac{1}{\rho(E)}\right), \tag{2}$$

where $\rho(E) \in \mathbb{C} \setminus \{0\}$ is a Floquet multiplier. Then $\Delta(E)$ is an entire function of order $\frac{1}{2}$ (see chapter 21 in [7]).

The spectrum $\sigma(H)$ of H is purely continuous and it consists of, at most, countably infinitely many analytical arcs whose endpoints E satisfy either $\Delta(E)^2 = 1$ or $\Delta'(E) = 0$ [8], where the prime denotes $\mathrm{d}/\mathrm{d}E$. Of course, when $V(x)$ is real-valued, these arcs lie on the real axis. So the spectrum consists of bands on the real axis.

We will make use of the following lemmas in investigating the shape of the spectrum $\sigma(H)$ when the potential $V(x)$ is complex-valued.

Lemma 1. *The following four assertions are equivalent.*

(i) $E \in \sigma(H)$.
(ii) $|\rho(E)| = 1$.
(iii) $\Delta(E)$ *is real and* $-1 \leqslant \Delta(E) \leqslant 1$.
(iv) *Equation* (1) *with* $E \in \mathbb{C}$ *has a non-constant bounded solution on* $\mathbb{R}$.

Proof. See, for example, chapters 1 and 5 in [6] for a proof. □

Lemma 2.

$$\sigma(H) \subset \{E \in \mathbb{C} \colon \mathrm{Re}(E) \geqslant M_1, M_2 \leqslant \mathrm{Im}(E) \leqslant M_3\},$$

where $M_1 = \inf_{x\in\mathbb{R}} \mathrm{Re}(V(x))$, $M_2 = \inf_{x\in\mathbb{R}} \mathrm{Im}(V(x))$ *and* $M_3 = \sup_{x\in\mathbb{R}} \mathrm{Im}(V(x))$.

Proof. Theorem 2.4.2 in [6] and lemma 1 above imply that $E \in \sigma(H)$ if and only if (1) has a non-constant solution $\psi(E, \cdot)$ with $\psi(E, \omega) = \mathrm{e}^{\mathrm{i}t}\psi(E, 0)$ and $\psi_x(E, \omega) = \mathrm{e}^{\mathrm{i}t}\psi_x(E, 0)$ for some $t \in \mathbb{R}$. Then we multiply (1) by $\overline{\psi}$, integrate over $[0, \omega]$ and use integration by parts to get

$$\int_0^\omega |\psi_x(E,x)|^2\,\mathrm{d}x + \int_0^\omega V(x)|\psi(E,x)|^2\,\mathrm{d}x = E\int_0^\omega |\psi(E,x)|^2\,\mathrm{d}x.$$

Then the lemma is an easy consequence of this equation. □

We next describe the local shape of the spectrum $\sigma(H)$.

Theorem 3. *Let $V \in L^1_{loc}(\mathbb{R})$ be a complex-valued periodic function on $\mathbb{R}$ with period $\omega > 0$. Suppose that $\Delta'(E_0) \neq 0$.*

(i) *If $-1 < \Delta(E_0) < 1$, then the spectrum near E_0 is an analytic arc.*

(ii) *If $\Delta(E_0) = \pm 1$, then the spectrum near E_0 consists of an analytic arc, one of whose endpoints is E_0.*

Suppose that $\Delta'(E_0) = \Delta''(E_0) = \cdots = \Delta^{(k-1)}(E_0) = 0$ and $\Delta^{(k)}(E_0) \neq 0$ for some $E_0 \in \mathbb{C}$ and integer $k \geqslant 2$.

(iii) *If $-1 < \Delta(E_0) < 1$, then the spectrum $\sigma(H)$ near $E_0 \in \mathbb{C}$ consists of $2k$ analytic arcs that have a common endpoint E_0. Moreover, adjacent arcs meet at E_0 at an angle of π/k.*

(iv) *If $\Delta(E_0) = \pm 1$, then the spectrum $\sigma(H)$ near $E_0 \in \mathbb{C}$ consists of k analytic arcs that have a common endpoint E_0. Moreover, adjacent arcs meet at E_0 at an angle of $2\pi/k$.*

Proof. Since $\Delta(E)$ is an entire function (of order $\frac{1}{2}$) of $E \in \mathbb{C}$, we have that

$$\begin{aligned}\Delta(E) &= \Delta(E_0) + \sum_{j=k}^{\infty} \frac{\Delta^{(j)}(E_0)}{j!}(E - E_0)^j \\ &= \Delta(E_0) + \frac{\Delta^{(k)}(E_0)}{k!}(E - E_0)^k\,[1 + O(|E - E_0|)]\,, \qquad (3)\end{aligned}$$

where $k \in \mathbb{N}$ is the positive integer such that $\Delta^{(k)}(E_0) \neq 0$, $\Delta^{(j)}(E_0) = 0$ for all $j = 1, 2, \ldots, k-1$.

Suppose that $\Delta(E_0)$ is real. Then $\Delta(E)$ is real if and only if $\sum_{j=k}^{\infty}(\Delta^{(j)}(E_0)/j!)(E - E_0)^j$ is real. Clearly,

$$\begin{aligned}\mathrm{Im}(\Delta(E)) &= \mathrm{Im}\left(\sum_{j=k}^{\infty} \frac{\Delta^{(j)}(E_0)}{j!}(E - E_0)^j\right) \\ &\underset{E\to E_0}{=} \mathrm{Im}\left(\frac{\Delta^{(k)}(E_0)}{k!}(E - E_0)^k[1 + O(|E - E_0|)]\right).\end{aligned}$$

So if $\Delta(E)$ is real with $|E - E_0|$ small, then

$$\arg(E - E_0) \underset{E\to E_0}{=} \frac{1}{k}[j\pi - \arg(\Delta^{(k)}(E_0))] + O(|E - E_0|), \qquad j = 1, 2, \ldots, 2k. \qquad (4)$$

Proof of (i) and (ii). Suppose that $\Delta'(E_0) \neq 0$. Then assertions (i) and (ii) are easy consequences of a fact from analytic function theory, that is, if $\Delta'(E_0) \neq 0$ then there is a neighbourhood $\mathcal{O}$ of E_0 in the complex E-plane such that $\Delta(E)|_{\mathcal{O}}$ has an analytic inverse function.

Proof of (iii) and (iv). Suppose that $\Delta'(E_0) = \Delta''(E_0) = \cdots = \Delta^{(k-1)}(E_0) = 0$ and $\Delta^{(k)}(E_0) \neq 0$ for some $E_0 \in \mathbb{C}$ and integer $k \geqslant 2$. Then we can deduce from (3) and (4) that when $\Delta(E_0)$ is real, $\Delta(E)$ near E_0 is real along $2k$ analytic arcs that have a common endpoint E_0. Also, by (4), one sees that adjacent arcs meet at E_0 at an angle of π/k. By lemma 1, all these arcs near E_0 belong to the spectrum $\sigma(H)$ if $-1 < \Delta(E_0) < 1$. This completes the proof of (iii). However, if $\Delta(E_0) = \pm 1$ then by lemma 1, every other arc belongs to the spectrum $\sigma(H)$ and, hence, the adjacent arcs that belong to the spectrum meet at E_0 at an angle of $2\pi/k$. This completes the proof of (iv). ☐

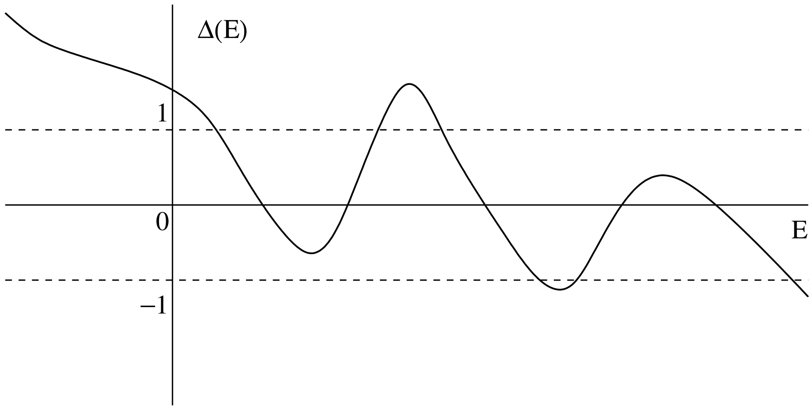

Figure 1. A sample graph of the Floquet discriminant $\Delta(E)$, $E \in \mathbb{R}$, when V is $\mathcal{PT}$-symmetric.

Remark. Batchenko and Gesztesy [9] studied quasi-periodic algebro-geometric KdV potentials and described the corresponding spectrum by using $\langle g(E,\cdot)^{-1}\rangle = (1/\omega)\int_0^{\omega}(1/g(E,x))\,\mathrm{d}x$, where $g(E,\cdot)$ is the diagonal Green's function of H. In the special periodic case, this reduces to $\langle g(E,\cdot)^{-1}\rangle = -(2/\omega)\ln(\Delta(E) + \sqrt{\Delta(E) - 1})$ (see equation (C.19) in [9]).

Next, we will show the existence of nonreal spectra of Hill operators with $\mathcal{PT}$-symmetric (i.e. $\overline{V(-x)} = V(x)$ for all $x \in \mathbb{R}$) periodic potentials V under some conditions. We will use this to interpret some numerical and analytic studies in [1–5].

Corollary 4. *Suppose that $\overline{V(-x)} = V(x)$ for all $x \in \mathbb{R}$, and suppose that V is periodic. If $\Delta(E)$ has a local maximum or minimum at some $E_0 \in \mathbb{R}$ with $-1 < \Delta(E_0) < 1$, then there must be some nonreal complex numbers E in the spectrum $\sigma(H)$.*

Proof. We first observe that when the periodic potential V is $\mathcal{PT}$-symmetric, $\psi(E, x)$ is a (bounded) solution of (1) corresponding to E if and only if $\overline{\psi(E, -x)}$ is a (bounded) solution of (1) corresponding to $\overline{E}$. Then, from lemma 1, one can show that the spectrum $\sigma(H) \subset \mathbb{C}$ is symmetric with respect to the real axis. Moreover, $\overline{\Delta(\overline{E})} = \Delta(E)$ and, hence, $\Delta(E)$ is real for all $E \in \mathbb{R}$.

Next, since $\Delta(E)$ has a local maximum or minimum at $E_0 \in \mathbb{R}$, we have $\Delta'(E_0) = 0$. Then since $\Delta(E)$ is an entire function of order $\frac{1}{2}$, there exists $k \in \mathbb{N}$ and $k \geqslant 2$ such that $\Delta'(E_0) = \Delta''(E_0) = \cdots = \Delta^{(k-1)}(E_0) = 0$ and $\Delta^{(k)}(E_0) \neq 0$. So by theorem 3, $2k$ analytic arcs meet at E_0. Since adjacent arcs meet at an angle of π/k, there exist some nonreal numbers E in the spectrum $\sigma(H)$. □

In [1–5], Hill operators with $\mathcal{PT}$-symmetric periodic potentials V are studied. More precisely, the potentials $V(x) = \mathrm{i}\sin^{2n+1} x$ are considered in [2], whereas $\mathcal{PT}$-symmetric potentials with some delta distributions are studied in [1, 3–5]. In [1–5], graphs of $\Delta(E)$, $E \in \mathbb{R}$, are presented similar to figure 1 and it is argued that some of the real energy bands appear and disappear under perturbations. It was also argued that some of the (anti)periodic band edges are absent due to the appearance and disappearance of real energy bands. However, they failed to point out the existence of nonreal spectra and nonreal band edges as a result of complex deformations of real intervals of the spectrum under perturbations.

The main point of the graph of $\Delta(E)$ in figure 1 is that it has local extrema in $(-1, 1)$. If this happens, then we know by corollary 4 that nonreal spectra exist. Figure 2 shows a *possible*

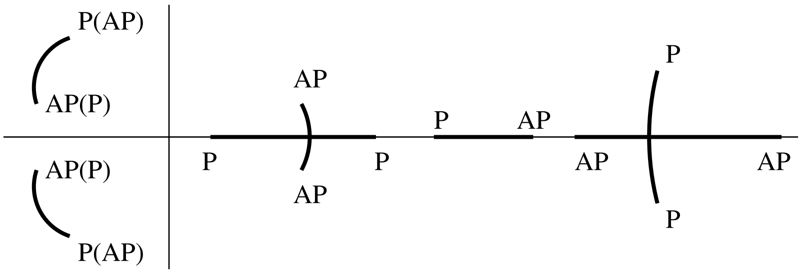

Figure 2. A *possible* spectrum $\sigma(H)$ denoted by thick curves, where anti-periodic and periodic band edges are labelled as AP and P, respectively. The pair of curves on the far left is *possible* spectra away from the real axis.

spectrum of H, corresponding to $\Delta(E)$ in figure 1, where the pair of curves on the far left is away from the real line. This happens after a real energy band moves off the real axis under perturbations and before a new energy band appears on the real axis. So the (anti)periodic band edges are not absent in this case, but they become nonreal.

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